Necessary and sufficient condition for stabilizability of discrete-time linear switched systems: a set-theory approach

Mirko Fiacchini, Marc Jungers

Necessary and sufficient condition for stabilizability of discrete-time linear switched systems: a set-theory approach [⋆]

Mirko Fiacchini [*], **Marc Jungers** [**,***],

[*] *GIPSA-lab, Grenoble Campus, 11 rue des Mathématiques, BP 46, 38402 Saint Martin d'Hères Cedex, France*
mirko.fiacchini@gipsa-lab.fr
[**] *Université de Lorraine, CRAN, UMR 7039, 2 avenue de la forêt de Haye, Vandœuvre-lès-Nancy Cedex, 54516, France*
marc.jungers@univ-lorraine.fr
[***] *CNRS, CRAN, UMR 7039, France*

Abstract: In this paper, the stabilizability of discrete-time linear switched systems is considered. Several sufficient conditions for stabilizability are proposed in the literature, but not a necessary and sufficient one. The main contribution is a computation-oriented necessary and sufficient condition for stabilizability based on set-theory. Based on such a condition, an algorithm for computing the Lyapunov functions and a procedure to design the stabilizing switching control law are provided. The generic algorithm is based on the invariance of unions of compact, convex sets containing the origin and is applied to numerical examples using ellipsoids and polytopes. It will be shown in particular that no assumption is made on the existence of a Schur convex combination of the matrices, assumption on which the Lyapunov-Metzler inequalities approach is based. Further discussions with respect to the literature and concluding remarks are also proposed.

Keywords: Switched linear systems; set-theory; stabilization policy; invariance.

1. INTRODUCTION

Switched systems are systems for which the current dynamic, specified by the so-called switching law, belongs at each instant to a finite set of modes (see Liberzon (2003)). These last decades, a large literature has been devoted to study switched systems for practical reasons: they model complex systems like embedded ones; and for theoretical reasons: their behavior and associated properties like their stability are neither intuitive nor trivial, as it has been emphasized in Liberzon and Morse (1999). Due to the large variety of assumptions related to the switching law, several frameworks are distinguished. The most common approaches consider the switching law as a perturbation or as a part of the control inputs.

When the switching law is a perturbation, that is an arbitrary function, sufficient but conservative conditions to ensure the stability have been provided (see for overviews Lin and Antsaklis (2009); Sun and Ge (2011)), with common Lyapunov function, Lie algebra and differential (or difference) inclusions (Gurvits, 1995; Liberzon et al., 1999; Agrachev and Liberzon, 2001), multiple Lyapunov functions (Branicky, 1998), switched quadratic Lyapunov functions (Daafouz et al., 2002). In addition several refinements have been proposed in order to obtain necessary and sufficient conditions for stability of switched systems. Among these conditions, one can cite the joint spectral radius approach (Bauer et al., 1993; Lin and Antsaklis, 2004; Jungers, 2009); the formulation of a polyhedral Lyapunov function (Molchanov and Pyatnitskiy, 1989) or a path-depend switched Lyapunov one (Lee and Dullerud, 2007). It should be also mentioned that for specific classes of switched systems, necessary and sufficient conditions could be obtained: for two-dimensional systems (Boscain, 2002), for positive ones (Gurvits et al., 2007).

In the case where the switching law is a part of the control inputs, sufficient conditions for stabilizability have been provided, mainly by using a *min-switching* policy (Liberzon, 2003, Chapter 3) introduced in Wicks et al. (1994), developed in (Kruszewski et al., 2011) via BMI and leading to Lyapunov-Metzler inequalities (Geromel and Colaneri, 2006). Based on the set-induced Lyapunov functions introduced in Blanchini (1995), sufficient conditions for stabilization or more precisely uniformly ultimate boundedness has been proposed for uncertain switched linear systems in Lin and Antsaklis (2003). Nevertheless to the best knowledge of the authors, there does not exist in the literature necessary and sufficient conditions for the stabilizability of discrete-time switched linear system.

The aim of this paper is to provide necessary and sufficient conditions for stabilizability of linear discrete time switched systems. The set-theory will be used and will offer a numerically sound algorithm to check the stabilizability and also the switching control law stabilizing the switched system.

The outline of the paper follows. In Section 2, preliminaries and tools issued from the set-theory are presented. Results on stability for arbitrary switching laws are recalled in Section 3. Necessary and sufficient conditions for stabilizability of switched systems are provided in Section 4. The efficiency and suitability of our approach are underlined on academic examples in Section 5, before concluding remarks in Section 6.

[⋆] Corresponding author M. Fiacchini.

Notation: The set of positive integers smaller than or equal to the integer $n \in \mathbb{N}$ is denoted as $\mathbb{N}_n$, i.e. $\mathbb{N}_n = \{x \in \mathbb{N} : 1 \leq x \leq n\}$. Given $D, E \subseteq \mathbb{R}^n$, $\alpha \geq 0$ and $M \in \mathbb{R}^{m \times n}$, define $D + E = \{z = x + y \in \mathbb{R}^n : x \in D,\ y \in E\}$, define $D - E = \{x \in \mathbb{R}^n : x + E \subseteq D\}$, $\alpha D = \{\alpha x \in \mathbb{R}^n : x \in D\}$ and $MD = \{Mx \in \mathbb{R}^m : x \in D\}$. Given a set $D \subseteq \mathbb{R}^n$, $\mathrm{co}(D)$ denoted the convex hull of D, $\mathrm{int}(D)$ its interior and ∂D its boundary. The set $\mathbb{B}^n$ is the unitary Euclidean ball in $\mathbb{R}^n$. The i-th element of a finite sets of matrices is denoted as A_i, of a set of sets as Ω^i.

2. PRELIMINARIES

Consider the discrete-time autonomous switched system

$$x_{k+1} = A_{\sigma(k)} x_k, \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state at time $k \in \mathbb{N}$ and $\sigma : \mathbb{N} \to \mathbb{N}_q$ is the switching law that, at any instant, selects the transition matrix among the finite set $\{A_i\}_{i \in \mathbb{N}_q}$, with $A_i \in \mathbb{R}^{n \times n}$ for all $i \in \mathbb{N}_q$. Given the initial state x_0 and a switching law $\sigma(\cdot)$, we denote with $x_N^\sigma(x_0)$ the state of the system (1) at time N starting from x_0 by applying the switching law $\sigma(\cdot)$. In some cases σ can be a function of the state, for instance in the case of switching control law, as shown later.

A concept widely employed in the context of set-theory and invariance is the C-set, see Blanchini (1991, 1995); Blanchini and Miani (2008). A C-set is a compact, convex set containing the origin in its interior. We define an analogous concept usefull for our purpose. For this, we first recall that a set Ω is a star-convex set if there exists $x^0 \in \Omega$ such that every convex combination of x and x^0 belongs to Ω for every $x \in \Omega$.

Definition 1. A set $\Omega \subseteq \mathbb{R}^n$ is a C*-set if it is compact, star-convex with respect to the origin and $0 \in \mathrm{int}(\Omega)$.

Define also the analogous of the gauge function of a C*-set as

$$\Psi_\Omega(x) = \min_\alpha \{\alpha \in \mathbb{R} : x \in \alpha\Omega\}, \tag{2}$$

for the C*-set $\Omega \subseteq \mathbb{R}^n$. In what follows, we will refer to $\Psi_\Omega(x)$ as the Minkowski function of Ω at x, with a slight abuse as the Minkowski function is usually defined for C-sets (or symmetric C-sets), Rockafellar (1970); Schneider (1993); Blanchini and Miani (2008).

Some basic properties of the C*-sets and their Minkowski functions are listed below. The proof is avoided, since they follow directly from the definition.

Property 1. Any C-set is a C*-set. Given a C*-set $\Omega \subseteq \mathbb{R}^n$, we have that $\alpha\Omega \subseteq \Omega$ for all $\alpha \in [0,1]$, and the Minkowski function $\Psi_\Omega(\cdot)$ is: homogenous of degree one, i.e. $\Psi_\Omega(\alpha x) = \alpha \Psi_\Omega(x)$ for all $\alpha \geq 0$ and $x \in \mathbb{R}^n$; positive definite; defined on $\mathbb{R}^n$ and radially unbounded.

Then, given a C*-set $\check{\Omega}$, its Minkowski function is a Lyapunov function if there exists $N \geq 1$ and a switching law defined on $\mathbb{R}^n$ such that its value is not increasing and it decreases after N steps for all $x \in \mathbb{R}^n$. Although this is not the classical definition of Lyapunov functions, it can be proved that there exists a Lyapunov function if and only if there is a function of this kind. Notice that this is equivalent to impose that the smaller level set containing $x_N^\sigma(x)$ is contained in the interior of the smaller one containing x for all x, which is equivalent to contractivity of $\check{\Omega}$ after N steps.

3. RECALL OF THE ARBITRARY SWITCHING LAW FRAMEWORK

In this section, we recall in Theorem 1 necessary and sufficient conditions for the stability of a switched linear system with arbitrary switching law σ. Several statements are then declined in Proposition 1 and links with the literature are discussed.

Theorem 1. (Molchanov and Pyatnitskiy (1989)). There exists a Lyapunov function for the switching system (1) if and only if there exists a C-set $\hat{\Omega} \subseteq \mathbb{R}^n$ and a scalar $\lambda \in [0,1)$ such that

$$A_i \hat{\Omega} \subseteq \lambda \hat{\Omega}, \quad \forall i \in \mathbb{N}_q. \tag{3}$$

Proof: The proof is inspired by the one of Lemma 4.1 in Blanchini (1995) and extended to the case of switched systems without loss of genericity. It is based on the linearity with respect to the state of the system (1) and on the convexification of the set induced by the stability definition of the system. ■

The condition (3) could be reformulated in several forms, which are more suitable for the switched nature of the system (1).

Proposition 1. The three following statements are equivalent:

a) There exists a C-set $\hat{\Omega} \subseteq \mathbb{R}^n$, such that

$$A_i \hat{\Omega} \subseteq \lambda \hat{\Omega}, \quad \forall i \in \mathbb{N}_q. \tag{4}$$

b) There exist q C-sets $\hat{\Omega}_i \subseteq \mathbb{R}^n$, with $i \in \mathbb{N}_q$, such that

$$A_i \hat{\Omega}_i \subseteq \lambda \cap_{j \in \mathbb{N}_q} \hat{\Omega}_j, \quad \forall i \in \mathbb{N}_q. \tag{5}$$

c) There exist q C-sets $\hat{\Omega}_j \subseteq \mathbb{R}^n$, with $i \in \mathbb{N}_q$, such that

$$A_i \hat{\Omega}_i \subseteq \lambda \hat{\Omega}_j, \quad \forall (i,j) \in \mathbb{N}_q \times \mathbb{N}_q. \tag{6}$$

Proof: The proof is made circularly. The implication a)⇒b) is true by choosing $\hat{\Omega}_i = \hat{\Omega}$, $\forall i \in \mathbb{N}_q$. The inclusion $\bigcap_{j \in \mathbb{N}_q} \hat{\Omega}_j \subseteq \hat{\Omega}_i$, $\forall i \in \mathbb{N}_q$ leads to b)⇒c). The implication c)⇒a) is obtained by considering $\hat{\Omega} = \bigcap_{j \in \mathbb{N}_q} \hat{\Omega}_j$, which is a C-set because $\hat{\Omega}_i$ are C-sets. More precisely, the inclusion (6) being true for all $j \in \mathbb{N}_q$, we have $A_i \hat{\Omega}_i \in \lambda \bigcap_{j \in \mathbb{N}_q} \Omega_j$ and $A_i \bigcap_{j \in \mathbb{N}_q} \Omega_j \in \lambda \bigcap_{j \in \mathbb{N}_q} \Omega_j$. ■

The relation (4) is more convenient for theoretical and computational aspects because it is closer to the set-induced Lyapunov function proposed in Blanchini (1995). When $\hat{\Omega}$ is assumed to be an ellipsoid, we obtain sufficient conditions for stability with a common quadratic Lyapunov function. We recover the result of Molchanov and Pyatnitskiy (1989) concerning necessary and sufficient conditions with a polyhedral Lyapunov function by assuming that $\hat{\Omega}$ is a polytope, due to the fact that a C-set admits and arbitrarly close polytopic approximation. When Ω_i, $i \in \mathbb{N}_q$ are ellipsoids, the relation (6) writes as the sufficient conditions for stability in the framework of quadratic switched Lyapunov functions (Daafouz et al., 2002). Finally the relation (5) is adapted to design an algorithm based on pre-image modal operators. This last approach will be privileged in the following for the case of switching control law.

4. SWITCHING CONTROL LAW

It is proved in Molchanov and Pyatnitskiy (1989) that for an autonomous linear switched system (called therein difference inclusion), the origin is asymptotically stable if and only if there exists a polyhedral Lyapunov function, see also Blanchini (1995); Lin and Antsaklis (2009). It can be proved that analogous results can be stated in the case that the switching sequence is supposed to be a properly chosen selection, that is considering it as a control law.

We recall that in the switching stabilization literature, the system (1) is asymptotically stabilizable if there exists a switching law and a continuous positive definite and radially unbounded non-increasing function converging to zero when the law is applied. Hence, the function is a Lyapunov function and standard reasoning for guaranteeing asymptotically stability hold for the resulting time-varying system. The switching law will belong to the class of state-dependent one, that is

$$\sigma(k) = g(x_k), \tag{7}$$

where $g : \mathbb{R}^n \mapsto \mathbb{N}_q$. With a slight abuse of notation we define in the sequel the state-dependent switching law as $\sigma(k) = \sigma(x_k)$.

Assumption 1. The matrices A_i, with $i \in \mathbb{N}_q$, are non-singular.

Remark 1. Notice that this assumption is not restrictive at all. In fact, the stable eigenvalues of the matrices A_i are beneficial from the stability point of view of the switched systems and poles in zero are related to the more contractive dynamics. Moreover, the results presented in the following can be extended to the general case with appropriate considerations. Finally, recall that sampled linear systems do not present poles in the origin and then real systems satisfy Assumption 1.

Consider the following algorithm:

Algorithm 1. Computation of a contractive C*-set for the system (1) such that Assumption 1 holds.

- **Initialization**: given the C*-set $\Omega \subseteq \mathbb{R}^n$, define $\Omega_0 = \Omega$ and $k = 0$;
- **Iteration** for $k \geq 0$:
$$\begin{aligned} \Omega_{k+1}^i &= A_i^{-1}\Omega_k, \quad \forall i \in \mathbb{N}_q, \\ \Omega_{k+1} &= \bigcup_{i\in\mathbb{N}_q} \Omega_{k+1}^i; \end{aligned} \tag{8}$$
- **Stop** if $\Omega \subseteq \operatorname{int}\Big(\bigcup_{j\in\mathbb{N}_{k+1}} \Omega_j\Big)$; denote $\check{N} = k+1$ and
$$\check{\Omega} = \bigcup_{j\in\mathbb{N}_{\check{N}}} \Omega_j. \tag{9}$$

From the geometrical point of view, notice that the set Ω_{k+1}^i is the set of points mapped in Ω_k through A_i. Then Ω_{k+1} are those points $x \in \mathbb{R}^n$ for which there exists a selection $i(x) \in \mathbb{N}_q$ such that $A_{i(x)}x \in \Omega_k$. Therefore, Ω_k is the set of points that can be driven in Ω in k step and hence $\check{\Omega}$ the set of those which can reach Ω in $\check{N}$ or less steps, by an adequate switching law.

Remark 2. The symbol A_i^{-1} should be intended to denote, with a slight abuse of notation, the operator that associates to a set its inverse image, rather than the inverse of matrix A_i. That is, given the set $D \subseteq \mathbb{R}^n$ one have

$$A_i^{-1}D = \{x \in \mathbb{R}^n : A_i x \in D\}.$$

Indeed, it is worth pointing out that the inverse image of a set exists and can be computed also for linear transformations given by non-invertible matrices. This has to remarked because the algorithm applies also for the general case for which Assumption 1 is not satisfied, although it is not the case so far.

Proposition 2. The sets Ω_k^i and Ω_k with $i \in \mathbb{N}_q$ and for all $k \geq 0$ are C*-sets.

Proof: Clearly Ω_0 is a C*-set. It is sufficient to prove that $A^{-1}D$ and $D \cup E$ are C*-sets, for all nonsingular $A \in \mathbb{R}^n$ and every C*-sets D and E to prove the results by induction. By definition $\alpha x \in D$ for all $x \in D$ and $\alpha \in [0,1]$. Then given $\alpha \in (0,1]$ we have

$$\begin{aligned} \alpha A^{-1}D &= \{\alpha x \in \mathbb{R}^n : Ax \in D\} = \{y \in \mathbb{R}^n : Ay \in \alpha D\} \subseteq \\ &\subseteq \{x \in \mathbb{R}^n : Ax \in D\} = A^{-1}D, \end{aligned}$$

since D is a C*-set. For $\alpha = 0$, $\alpha A^{-1}D = \{0\} \subseteq A^{-1}D$, trivially. Then $A^{-1}D$ is a star-convex set and it is also compact from Assumption 1. The fact that it contains the origin in its interior follows from the fact that A_i^{-1} are continuous operators under Assumption 1. Then $A^{-1}D$ is a C*-set. The property on the union follows from the definition of C*-set. ■

It can be proved that Algorithm 1 provides a C*-set $\check{\Omega}$ contractive in $\check{N}$ steps, for every initial C*-set $\Omega \in \mathbb{R}^n$, if and only if the switching system (1) is asymptotically stable. Such necessary and sufficient condition, which is the main contribution of the paper, is stated in the theorem below.

Theorem 2. There exists a Lyapunov function for the switching system (1) if and only if the Algorithm 1 ends with finite $\check{N}$.

Proof: Necessity follows from the fact that, if the algorithm ends with finite $\check{N}$, then $\check{\Omega}$ induces a Lypaunov function. Indeed, $\check{\Omega}$ being a C*-set from Proposition 2, its Minkowski function is defined. Moreover, considering

$$\check{\lambda} = \check{\lambda}(\Omega) = \min_{\lambda}\{\lambda \geq 0 : \Omega \subseteq \lambda\check{\Omega}\}, \tag{10}$$

we have that $\check{\lambda} < 1$, since $\Omega \subseteq \operatorname{int}(\check{\Omega})$ and $\check{\Omega}$ is a C*-set. Since by construction $\check{\Omega}$ is the set of points x such that $x_k^\sigma(x)$ are in Ω for $k = k(x) \leq \check{N}$ and an appropriate switching sequence, then we have

$$x_{k(x)}^\sigma(x) \in \Omega \subseteq \check{\lambda}\check{\Omega}, \tag{11}$$

for all $x \in \check{\Omega}$ and in particular for $x \in \partial\check{\Omega}$. This means that there exist a switching $\sigma(x)$ and $k(x) \leq \check{N}$ such that

$$\Psi_{\check{\Omega}}(x_{k(x)}^\sigma(x)) \leq \check{\lambda}\Psi_{\check{\Omega}}(x), \tag{12}$$

for all $x \in \partial\check{\Omega}$. Then the value of the Minkowski function decreases after $k(x)$ steps, for all x on the boundary. Moreover, it does not increase, for all $j \leq k(x)$. In fact, given $x \in \partial\check{\Omega}$, the elements $x_j^\sigma(x)$ can be stirred in Ω in $k(x) - j$ steps for all $j \leq k(x)$, being elements of the same sequence whose last element is in Ω. This means that $x_j^\sigma(x) \in \check{\Omega}$ and then

$$\Psi_{\check{\Omega}}(x_j^\sigma(x)) \leq \Psi_{\check{\Omega}}(x), \quad \forall j \in \mathbb{N}_{k(x)}. \tag{13}$$

for all $x \in \partial\check{\Omega}$. Then for every $x \in \partial\check{\Omega}$ there exists a switching sequence of length $k(x)$ such that the $\Psi_{\check{\Omega}}$ is not increasing for the first $k(x) - 1$ steps and it decreases of at least a proportional value $\check{\lambda}$ at the instant $k(x)$, from (12) and (13).

Since every x is on the boundary of a level set of $\Psi_{\check{\Omega}}(x)$, in particular $x \in \partial\big(\Psi_{\check{\Omega}}(x)\check{\Omega}\big)$, and from the homogeneity of the Minkowski function and the linearity of the switched system, we have that (12) and (13) hold for every $x \in \mathbb{R}^n$. Thus from (12) and (13) valid on the whole state space, we have

$$\Psi_{\check{\Omega}}(x_{\check{N}}^\sigma(x)) \leq \check{\lambda}\Psi_{\check{\Omega}}(x), \tag{14}$$

for all $x \in \mathbb{R}^n$ which proves that $\Psi_{\check{\Omega}}$ is a Lyapunov function. Then, if the Algorithm 1 ends with finite $\check{N}$, a Lyapunov function exists, in particular $\Psi_{\check{\Omega}}(x)$.

To prove sufficiency, we suppose that there exists a Lyapunov function for the switched linear system (1) and we demonstrate that the Algorithm 1 ends with finite $\check{N}$. By definition, there exist a switching law $\sigma(x)$, a value $N \in \mathbb{N}$ and a continuous positive definite function $V : \mathbb{R}^n \to \mathbb{R}$ such that for every x we have $V(x_N^\sigma(x)) < V(x)$. Consider the set

$$\Omega^V = \{x \in \mathbb{R}^n : V(x) \leq 1\}, \tag{15}$$

which is closed from continuity of V and bounded from its radially unboundedness. Hence Ω^V is compact and $0 \in \mathrm{int}(\Omega^V)$, since V is continuous and positive definite. Thus for every C*-set Γ, there exists $\varepsilon > 0$ such that the C*-set $\varepsilon\Gamma$ satisfies $\varepsilon\Gamma \in \mathrm{int}(\Omega^V)$. Posing $\Omega = \varepsilon\Gamma$ in Algorithm 1, we have $\Omega \subseteq \mathrm{int}(\Omega^V)$. From the gobally asymptotic stability of the system (1), there exist a switching law $\sigma(x)$ defined on $\mathbb{R}^n$ and a finite $N^V \in \mathbb{N}$ such that for all $x \in \Omega^V$ there exists $k(x) \leq N^V$ for which

$$x_{k(x)}^\sigma(x) \in \Omega.$$

Consider Ω_{N^V} obtained by applying the Algorithm 1 with Ω defined above, supposing that the stop condition has not been satisfied, otherwise the result would be directly proved. Since the set Ω_j is the set of states that can be stirred in Ω in j steps, then $\Omega^V \subseteq \bigcup_{j\in\mathbb{N}_{N^V}} \Omega_j$ and then we have

$$\Omega \subseteq \mathrm{int}(\Omega^V) \subseteq \mathrm{int}\big(\bigcup_{j\in\mathbb{N}_{N^V}} \Omega_j\big),$$

which contradicts the fact that the stop condition has not been satisfied. Then the Algorithm 1 ends with finite $\check{N}$ with $\check{N} \leq N^V$. ∎

The fact that Algorithm 1 ends with finite $\check{N}$ is a necessary and sufficient condition for global asymptotic stability of the switched system (1). Moreover, Algorithm 1 provides a Lyapunov function and a stabilizing switching control law, or better a family of stabilizing control laws, for the linear switched system (1). Notice that the complexity of the algorithm, which we think should deserve further analysis, depends on the complexity intrinsic to the Lyapunov functions and then on the nature proper of the system.

Proposition 3. If Algorithm 1 ends with finite $\check{N}$ then $\Psi_{\check{\Omega}} : \mathbb{R}^n \mapsto \mathbb{R}$ is a Lyapunov function for the switched system (1) and given the set valued map

$$\check{\Sigma}(x) = \arg\min_{(i,k)}\{\Psi_{\Omega_k^i}(x) : i \in \mathbb{N}_q, k \in \mathbb{N}_{\check{N}}\} \subseteq \mathbb{N}_q \times \mathbb{N}_{\check{N}}, \tag{16}$$

any switching law defined as

$$(\check{\sigma}(x), \check{k}(x)) \in \check{\Sigma}(x), \tag{17}$$

is a stabilizing switching law and such that

$$\begin{aligned} &\Psi_{\check{\Omega}}(x_{\check{k}(x)}^{\check{\sigma}}(x)) \leq \check{\lambda}\Psi_{\check{\Omega}}(x), \\ &\Psi_{\check{\Omega}}(x_j^{\check{\sigma}}(x)) \leq \Psi_{\check{\Omega}}(x), \qquad \forall j \in \mathbb{N}_{\check{k}(x)}, \end{aligned}$$

with $\check{\lambda}$ as in (10).

Proof: The fact that $\Psi_{\check{\Omega}}(\cdot)$ is a Lyapunov function has been proved in the proof of necessity for Theorem 2. Denote $\alpha = \Psi_{\check{\Omega}}(x)$, to easy the notation. Then $x \in \partial(\alpha\check{\Omega})$ by definition. Moreover, from definition of $\check{\Omega}$, there are some values $(i,k) \in \mathbb{N}_q \times \mathbb{N}_{\check{N}}$ such that $x \in \partial(\alpha\Omega_k^i)$, since $\check{\Omega}$ is the union of Ω_k^i for all $i \in \mathbb{N}_q$ and $k \in \mathbb{N}_{\check{N}}$. Concerning the (i,k) for which $x \in \partial(\alpha\Omega_k^i)$ is not satisfied, we have that $x \notin (\alpha\Omega_k^i)$ and then $\Psi_{\Omega_k^i}(x) > \alpha$. This because x is either on the boundary or in the complement of every $\alpha\Omega_k^i$, for all $i \in \mathbb{N}_q$ and $k \in \mathbb{N}_{\check{N}}$, otherwise α would not be the minimal value such that $x \in \alpha\check{\Omega}$. Then for every $i \in \mathbb{N}_q$ and $k \in \mathbb{N}_{\check{N}}$ we have that

$$\begin{cases} \Psi_{\Omega_k^i}(x) = \alpha, & \text{if } x \in \partial(\alpha\Omega_k^i), \\ \Psi_{\Omega_k^i}(x) > \alpha, & \text{if } x \notin (\alpha\Omega_k^i). \end{cases}$$

Remind that by construction Ω_k^i is the set that can be stirred in Ω, and then also in the contracted set $\check{\lambda}\check{\Omega}$, in k steps by means of a sequence of modes whose first element is i. Moreover, as demonstrated in the proof of Theorem 2, the Minkowski function does not increase along the first $k-1$ elements of the generated trajectory. Then from homogeneity of the Minkowski functions, the set $\Sigma(x)$ is composed by the pairs (i,k) where i is the first element of a control sequence $\sigma(x)$ that leads to have $x_k^\sigma(x) \in \check{\lambda}\alpha\check{\Omega}$ and $x_j^\sigma(x) \in \alpha\check{\Omega}$ for all $j \in \mathbb{N}_k$. As $(\check{\sigma(x)}, \check{k}(x))$ is a selection of the set $\Sigma(x)$, the result follows. ∎

It could be reasonable, to speed up the convergence, to select among the elements of $\Sigma(x)$, those whose k is minimal.

Corollary 1. If Algorithm 1 ends with finite $\check{N}$ then the switching law defined by (16) and (17) is such that

$$\Psi_{\check{\Omega}}(x_{p\check{N}}^{\check{\sigma}}(x)) \leq \check{\lambda}^p \Psi_{\check{\Omega}}(x), \tag{18}$$

for every $p \in \mathbb{N}$ and all $x \in \mathbb{R}^n$.

Proof: From Proposition 3 we have that, if Algorithm 1 ends with finite $\check{N}$, then there exist a switching law $\check{\sigma}(x)$ and the related $\check{k}(x) \leq \check{N}$ such that the Minkowski function of $\check{\Omega}$ does not increase for $k \leq \check{k}(x)$ and it decreases of a proportional value of $\check{\lambda}$ after $\check{k}(x)$ steps, for all $x \in \mathbb{R}^n$. Since $\check{k}(x) \leq \check{N}$, then the value of $\Psi_{\check{\Omega}}(x)$ decreases at least one time within the next $\check{N}$ steps, that means that

$$\Psi_{\check{\Omega}}(x_{\check{N}}^{\check{\sigma}}(x)) \leq \check{\lambda}\Psi_{\check{\Omega}}(x),$$

which implies (18) since the property applies over the whole space $\mathbb{R}^n$. ∎

Remark 3. It is worth pointing out that if the system is asymptotically stabilizable, then the algorithm ends with finite $\check{N}$ for all initial C*-set Ω. Clearly, the value of $\check{N}$ and the complexity of the set $\check{\Omega}$ depends on the choice of Ω. In particular, if Ω is the euclidean norm ball (or the union of full dimensional ellipsoids), the sets Ω_k^i and Ω_k, with $i \in \mathbb{N}_q$ and $k \in \mathbb{N}_{\check{N}}$, are union of ellipsoids, and so is $\check{\Omega}$. Then, the switching law computation reduces to check the minimal value of among $x^T P_j x$ with $j \in \check{M}$, where $\{P_j\}_{j\in\check{M}}$ are the $\check{M}$ positive definite matrices that define $\check{\Omega}$, with $\check{M} = \sum_{k\in\mathbb{N}_{\check{N}}} q^k = q + \cdots + q^{\check{N}} = (q^{\check{N}+1} - q)/(q-1)$. Moreover, if Ω is a (the union of) polytope contaning the origin in its interior, also Ω_k^i, Ω_k, with $i \in \mathbb{N}_q$ and $k \in \mathbb{N}_{\check{N}}$, and $\check{\Omega}$ are so. In this case, the switching law is obtained by evaluating the set linear inequalities defining those polytopes.

5. ILLUSTRATION

In order to illustrate the suitability of the algorithm, consider the example with $q = n = 2$:

$$A_1 = \begin{bmatrix} 1.2 & 0 \\ -1 & 0.8 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.6 & -2 \\ 0 & -1.2 \end{bmatrix}.$$

Both the matrices A_1 and A_2 are not Schur, which implies that this is not possible to stabilize the system (1) with a constant switching law. In order to apply the Algorithm 1, we have to choose a particular initial C*–set Ω. Firstly we consider $\Omega = \mathbb{B}^2$. The induced sets Ω_k, $k \in \mathbb{N}$ will be thus unions of ellipsoids. The result at the first step is depicted in Figure 1, left.

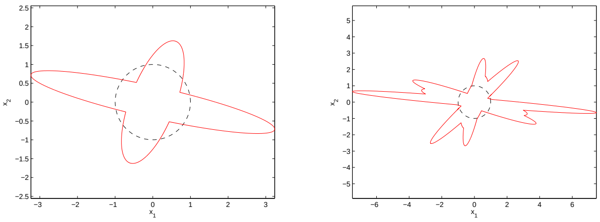

Fig. 1. Ball $\mathbb{B}^2$ in dashed line and induced sets Ω_1 and $\bigcup_{k\in\mathbb{N}_2}\Omega_k$ in solid line.

Ω_1 is the union of two ellipsoids ($A_1^{-1}\mathbb{B}^2$ and $A_2^{-1}\mathbb{B}^2$). It is clear that $\mathbb{B}^2$ does not belong to Ω_1. The next step of the algorithm leads to a set $\bigcup_{k\in\mathbb{N}_2}\Omega_k$ given by the union of six ellipsoids ($A_i^{-1}\mathbb{B}^2$ with $i\in\mathbb{N}_2$ and $A_j^{-1}A_i^{-1}\mathbb{B}^2$, for all $(i,j)\in\mathbb{N}_2\times\mathbb{N}_2$). Since $\mathbb{B}^2$ does not belong to $\bigcup_{k\in\mathbb{N}_2}\Omega_k$, see Figure 1 right, the termination condition is not satisfied. The algorithm stops at the fourth iteration. The zoom in Figure 2 emphasizes that $\mathbb{B}^2$ is included in $\bigcup_{k\in\mathbb{N}_4}\Omega_k$.

A stabilizing switching law, satisfying the relation (17) is given in Figure 3 for the initial condition $x_0=(-3,3)^T$. The Lyapunov function converges to zero (Figure 3). It is also noteworthy that the Lyapunov function is not a decreasing function, but only a non-increasing one which is strictly decreasing at least every four (the number of steps of the algorithm) instants, as proved in the main result.

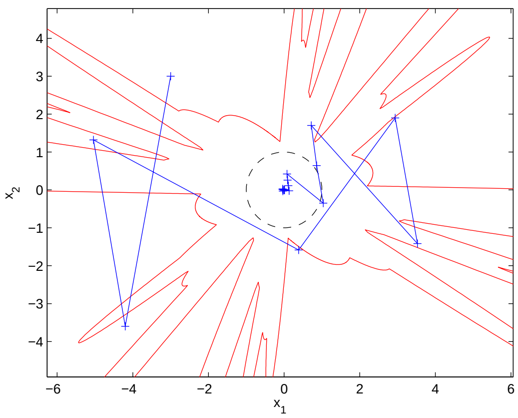

Fig. 2. Ball $\mathbb{B}^2$ in dashed line and $\bigcup_{k\in\mathbb{N}_4}\Omega_k$ in solid one. Trajectory starting from $x_0=(-3,3)^T$ in starry line.

The algorithm could also be applied by considering other initial C*–set Ω. For instance, we consider the unit square. The algorithm terminates at the fourth step. The related $\bigcup_{k\in\mathbb{N}_4}\Omega_k$ is depicted in Figure 4. The trajectory starting from $x_0=(-3,3)^T$ is also plotted in this figure.

As a second example, consider for $q=n=2$:

$$A_1=\begin{bmatrix}0 & -1.01\\ 1 & -1\end{bmatrix},\quad A_2=\begin{bmatrix}0 & -1.01\\ 1 & -0.5\end{bmatrix}.$$

Due to the structure of A_1 and A_2, the product of eigenvalues of every convex combination of these both matrices is equal to 1.01. That is every convex combination of A_1 and A_2 is not Schur. The technique based on Lyapunov-Metzler inequalities is then not applicable. Nevertheless this switched system is stabilizable. Our algorithm stops at the third step. It is shown in Figure 5 that $\mathbb{B}^2\subseteq\bigcup_{k\in\mathbb{N}_3}\Omega_k$. The Lyapunov function and the switching law are given in Figure 6. Notice that also in this case, the Lyapunov function decreases after $\check{N}$ steps.

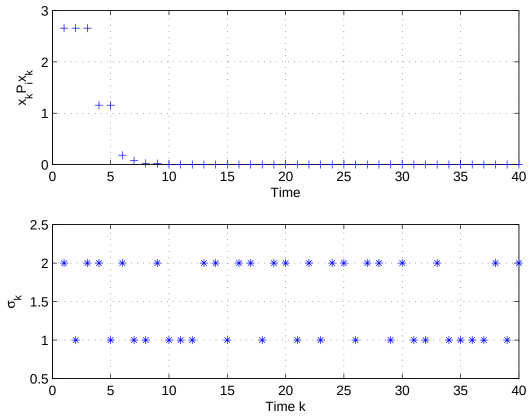

Fig. 3. Lyapunov function and switching control laws in time.

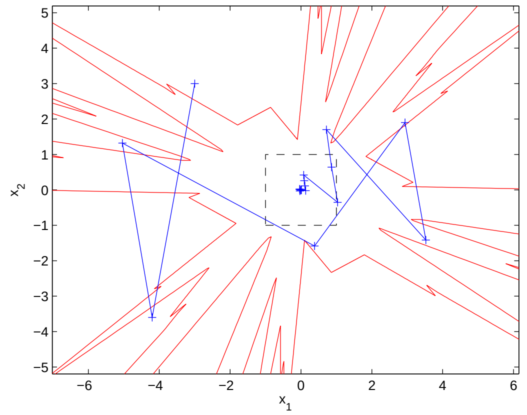

Fig. 4. Unitary square in dashed line and $\bigcup_{k\in\mathbb{N}_2}\Omega_k$ in solid one. Trajectory starting from $x_0=(-3,3)^T$ in starry line.

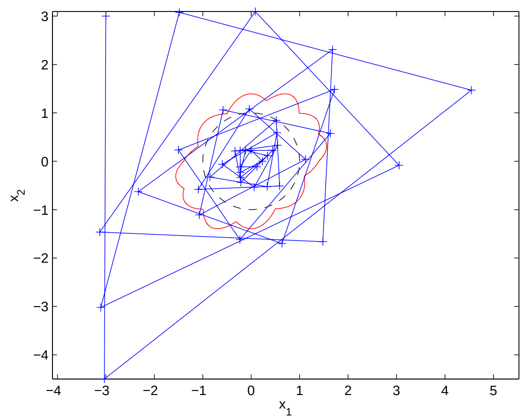

Fig. 5. Ball $\mathbb{B}^2$ in dashed line and $\bigcup_{k\in\mathbb{N}_3}\Omega_k$ in solid one. Trajectory starting from $x_0=(-3,3)^T$ in starry line.

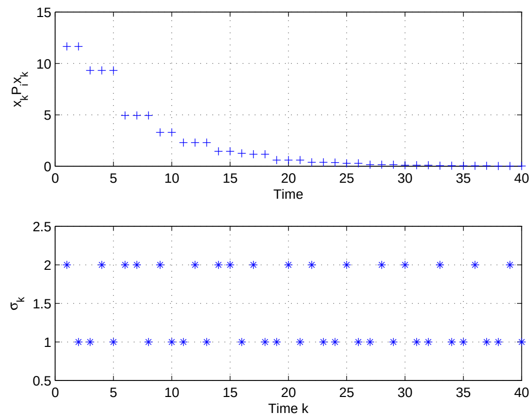

Fig. 6. Lyapunov function and switching control laws in time.

ACKNOWLEDGEMENTS

This work was partially supported by ANR project ArHyCo, Programme ”Systèmes Embarqués et Grandes Infrastructures” - ARPEGE, contract number ANR-2008 SEGI 004 01- 30011459. The research leading to these results has received funding from the European Community’s Seventh Framework Programme (FP7/2007-2013) under grant agreement n° 257462: HYCON2 Network of Excellence ”Highly-Complex and Networked Control Systems”. The authors would like to thank Professor Jamal Daafouz for helpful discussions concerning this paper.

6. CONCLUSION

The issue of the stabilizability of a switched discrete-time linear autonomous system has been studied in this paper. Via a set-theory approach, necessary and sufficient conditions for the stabilizability have been provided. These conditions are based on an algorithm using pre-image modal operators over compact, star-convex sets containing the origin in their interior, which provides in addition the switching laws stabilizing the switched system. The method is applied to numerical examples by employing ellipsoids or polytopes as the initial sets. Our approach allows moreover to stabilize counter-examples of Lyapunov-Metzler approach based on the existence of a Schur convex combination of the matrices. Several academic illustrations are proposed to strengthen the discussions and to emphasize the efficiency of our approach.

REFERENCES

Agrachev, A.A. and Liberzon, D. (2001). Lie-algebraic stability criteria for switched systems. *SIAM J. Control Optim.*, 40, 253–270.

Bauer, P.H., Premaratne, K., and Durán, J. (1993). A necessary and sufficient condition for robust asymptotic stability of time-variant discrete systems. *IEEE Transactions on Automatic Control*, 38(9), 1427–1430.

Blanchini, F. (1991). Ultimate boundedness control for uncertain discrete-time systems via set-induced Lyapunov functions. In *Proceedings of the 30th IEEE Conference on Decision and Control*, 1755–1760 vol.2. Brighton, England.

Blanchini, F. (1995). Nonquadratic Lyapunov functions for robust control. *Automatica*, 31, 451–461.

Blanchini, F. and Miani, S. (2008). *Set-Theoretic Methods in Control*. Birkhäuser.

Boscain, U. (2002). Stability of planar switched systems: the linear single input case. *SIAM J. Control Optim.*, 41(1), 89–112.

Branicky, M. (1998). Multiple Lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE Transactions on Automatic Control*, 43, 475–582.

Daafouz, J., Riedinger, P., and Iung, C. (2002). Stability analysis and control synthesis for switched systems : A switched Lyapunov function approach. *IEEE Transactions on Automatic Control*, 47, 1883–1887.

Geromel, J.C. and Colaneri, P. (2006). Stability and stabilization of discrete-time switched systems. *International Journal of Control*, 79(7), 719–728.

Gurvits, L. (1995). Stability of discrete linear inclusion. *Linear Algebra and its Applications*, 231, 47–85.

Gurvits, L., Shorten, R., and Mason, O. (2007). On the stability of switched positive linear systems. *IEEE Transactions on Automatic Control*, 52(6), 1099–1103.

Jungers, R.M. (2009). *The Joint Spectral Radius: Theory and Applications*. Springer-Verlag., Berlin Heidelberg.

A. Kruszewski, R. Bourdais, and W. Perruquetti. Converging algorithm for a class of BMI applied on state-dependent stabilization of switched systems. *Nonlinear Analysis: Hybrid Systems*, 5:647–654, 2011.

Lee, J.W. and Dullerud, G.E. (2007). Uniformly stabilizing sets of switching sequences for switched linear systems. *IEEE Transactions on Automatic Control*, 52, 868–874.

Liberzon, D. (2003). *Switching in Systems and Control*, volume in series Systems and Control: Foundations and Applications. Birkhäuser, Boston, MA.

Liberzon, D., Hespanha, J.P., and Morse, A.S. (1999). Stability of switched linear systems: A Lie-algebraic condition. *Systems & Control Letters*, 37(3), 117–122.

Liberzon, D. and Morse, A. (1999). Basic problems in stability and design of switched systems. *IEEE Control Systems Magazine*, 19, 59–70.

Lin, H. and Antsaklis, P.J. (2003). Synthesis of uniformly ultimate boundedness switching laws for discrete-time uncertain switched linear systems. In *Proceedings of the 42nd IEEE Conference on Decision and Control*, 4806–4811.

Lin, H. and Antsaklis, P.J. (2004). Persistent disturbance attenuation properties for networked control systems. In *Proceedings of the 43rd IEEE Conference on Decision and Control*, 953–958.

Lin, H. and Antsaklis, P.J. (2009). Stability and stabilizability of switched linear systems: a survey of recent results. *IEEE Transaction on Automatic Control*, 54(2), 308–322.

Molchanov, A.P. and Pyatnitskiy, Y.S. (1989). Criteria of asymptotic stability of differential and difference inclusions encounterd in control theory. *Systems & Control Letters*, 13, 59–64.

Rockafellar, R.T. (1970). *Convex Analysis*. Princeton University Press, USA.

Schneider, R. (1993). *Convex bodies: The Brunn-Minkowski theory*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, England.

Sun, Z. and Ge, S.S. (2011). *Stability Theory of Switched Dynamical Systems*. Springer.

Wicks, M.A., Peleties, P., and DeCarlo, R.A. (1994). Construction of piecewise Lyapunov functions for stabilizing switched systems. In *Proceedings of the 33rd IEEE Conference on Decision and Control*, 3492–3497.